Combinatorial study of neighborhood assortative mixing in networks

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Networks represent the topological skeleton of complex systems that are formed by many interacting-elements. The understanding of these structures and their patterns of connections is crucial for comprehending the evolutionary, functional, and dynamical processes taking place in these systems [1].

It is well known that, generally, links do not connect nodes regardless of their characteristics. Assortativity is a global metric of a graph that characterizes the nodes's tendency to link to other nodes of similar (or different) type. Usually, this concept is applied to the *degree* of the nodes (i.e. the number of direct neighbours, *k*), and it is is quantified by the Pearson coefficient of the degree-degree correlation [2]. However, it may be applied to other characteristics (or properties, *R*) of a node as well (see Eq. 1 for an undirected and unweighted network).

$$r_R = \frac{\langle \mathbf{R} \,|\mathbf{A}|\, \mathbf{R}\rangle - (\langle \mathbf{1} \,|\mathbf{A}|\, \mathbf{1}\rangle)^{-1} (\langle \mathbf{R} \,|\mathbf{A}|\, \mathbf{1}\rangle)^2}{\langle \mathbf{R} \circ \mathbf{R} \,|\mathbf{A}|\, \mathbf{1}\rangle - (\langle \mathbf{1} \,|\mathbf{A}|\, \mathbf{1}\rangle)^{-1} (\langle \mathbf{R} \,|\mathbf{A}|\, \mathbf{1}\rangle)^2} \quad (1)$$

where $\mathbf{A}$ is the adjacency matrix of the network ($\mathbf{A}_{ij} = 1$ if nodes i and j are linked, and zero otherwise), $|\mathbf{R}\rangle$ is a vector of node properties, $|\mathbf{1}\rangle$ is an all-ones vector, and $\circ$ denotes the Hadamard product. Networks have apparent assortative (disassortative) mixing by R, if $r_R > 0$ ($r_R < 0$).

Assortativity can be used to explore other global relations between the network's nodes and their respective neighborhoods. The nodes's tendency to link may not depend on its nearest neighbors individual properties, but on a neighborhood's average property [3]. That tendency is denoted by *neighborhood assortativity*.

In order to explore this possibility in an analytic way, we have assigned a new property to the network's nodes: the neighborhood's local degree, R.

$$R_i = \sum_{N_i} k_j \longleftrightarrow |\mathbf{R}\rangle = \mathbf{A}^2 \,|\mathbf{1}\rangle \quad (2)$$

where R_i is the i-node's local degree, k_j is the j-node's degree, and N_i is the set of nearest neighbors of node i.

By doing so, first of all, we have found that some real and synthetic networks exhibit positive neigborhood assortativity ($r_R > 0$) and null degree assortativity ($r_k \approx 0$) (see Figure 1 for an easy to visualize example). In fact, neighborhood assortativity is not a rare property in networks. For instance, 38% of the 11117 connected networks with 8 unlabelled nodes are assortative by neigborhood's local degree.

We have proved analytically that Eq. 1 can be expressed as the ratio of weighted sums of certain small sugraphs, such as those illustrated in Figure 2. In addition we have shown that the sign of Eq. 1, which determines the properties of assortativity/disassortativity, is determined by a weighted sum of such subgraphs. Furthermore, computer calculations over the collections of simple connected networks with 8, 9 and 10 unlabelled nodes (respectively) show that $r_R \geq r_k$. Therefore, in spite of its complexity, the neigborhood assortativity is strongly determined by the structural factors of the degree assortativiy: transitivity, intermodular connectivity and relative branching [1].

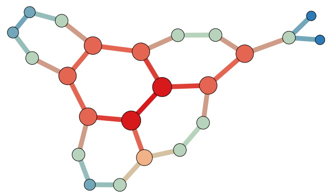

Figure 1: bio-MUTAG-g1 Network (included in the miscellaneous section of the networks available at [4]): $r_k = 0.08$, $r_R = 0.51$. Node's local degree is size-coded, from $R = 3$ to $R = 9$

Finally, it is worth noting that assortative mixing by vertex degree and by neigborhood's local degree is null for Erdös-Rényi random graphs ($r_k^{ER} \approx 0$, $r_R^{ER} \approx 0$). However, under this definition of neighborhood's local degree, the Barabási-Albert model produces assortative arrangements ($r_k^{BA} \approx 0$, $r_R^{BA} > 0$). So does the star graph model ($r_k^S = -1$, $r_R^S = 1$).

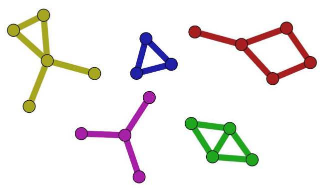

Figure 2: Examples of undirected subgraphs employed to calculate the neigborhood assortativity

Summarizing, this new approach can complement the information provided by the usual degree-degree correlations and the other graph metrics.

[1] E. Estrada, "Combinatorial study of degree assortativity in networks", Phys. Rev. E **84**, 047101 (2011).

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[4] Ryan A. Rossi and Nesreen K. Ahmed," The Network Data Repository with Interactive Graph Analytics and Visualization", in *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence*, (2015).