Equivalence between non-Markovian and Markovian dynamics in epidemic spreading processes

Michele Starnini[1], James P. Gleeson[2] and Marián Boguñá[1]

[1]Departament de Física de la Matèria Condensada, Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spain

[2]MACSI, Department of Mathematics and Statistics, University of Limerick, Ireland

For a long time, the Markovian assumption has been at the core of the modeling of stochastic dynamics governing many natural and technological systems, due to its mathematical tractability. Nevertheless, a new wealth of time-resolved data on different interactions, from human dynamics to natural phenomena, revealed that this modeling framework needs to be overcome. Here we introduce a general formalism to allow the steady state of non-Markovian (NM) processes on networks to be reduced to equivalent Markovian processes on the same substrates [1]. We consider the NM Susceptible-Infected-Susceptible (SIS) epidemic model [2] and show that its steady-state is equivalent to a Markovian one with an effective infection rate λ_{eff}, thus encoding all the NM effects into a single parameter. Interestingly, this result is independent of the underlying network topology.

In SIS dynamics, nodes are either susceptible or infected. An infected node decays spontaneously to the susceptible state after a random time t distributed as $\psi_R(t)$. A susceptible node, connected to a an infected one, becomes infected after a random time t has elapsed since the infection was initiated, with t distributed as $\psi_I(t)$. Distributions $\psi_R(t)$ and $\psi_I(t)$ allow us to evaluate the (time-dependent) hazard rates, defined as $\delta(t) = \psi_R(t)/\Psi_R(t)$ and $\lambda(t) = \psi_I(t)/\Psi_I(t)$, where $\Psi_R(t)$ and $\Psi_I(t)$ are the corresponding survival probabilities. In Markovian dynamics, both distributions are exponential and the corresponding hazard rates are constants.

Our mathematical formalism demonstrates the existence of an effective infection rate λ_{eff}, defined as

$$\lambda_{eff} \equiv \int_0^\infty \phi(\tau_{ji})\lambda(\tau_{ji})d\tau_{ji}, \qquad (1)$$

where $\phi(\tau_{ji}) \equiv \lim_{t\to\infty} \text{Prob}(\tau_{ji}; t|n_j = 1, n_i = 0)$ is the probability density of τ_{ji}, where τ_{ji} is the time elapsed since the start of the infection process from a node j to a node i, given that node i is susceptible and node j is infected and $\lambda(\tau_{ji})$ is averaged over all active links $i-j$ in the network. We are also able to derive an approximate analytic expression λ_{app}, in very good agreement with λ_{eff}.

We check the validity of the effective infection rates, λ_{eff} and λ_{app}, by means of extensive numerical simulations of the non-Markovian SIS dynamics. We consider a Markovian recovery process with rate δ, that is, $\psi_R(t) = \delta e^{-\delta t}$, and an infection process with a Weibull inter-event time distribution, $\psi_I(t) = \frac{\alpha_I}{b} \left(\frac{t}{b}\right)^{\alpha_I - 1} e^{-(t/b)^{\alpha_I}}$ with parameter α_I controlling the infection inter-event time distribution.

Fig. 1 shows the prevalence of the disease ρ^{st} at the steady state as a function of λ_{eff} and λ_{app}, for different network substrates: a two dimensional lattice, an Erdős Rényi (ER) graph, a random degree regular (RDR) network and a scale-free (SF) network with exponent $\gamma = 2.5$. One can see that different curves of the prevalence, corresponding to different forms of the infection inter-event time distribution, col-
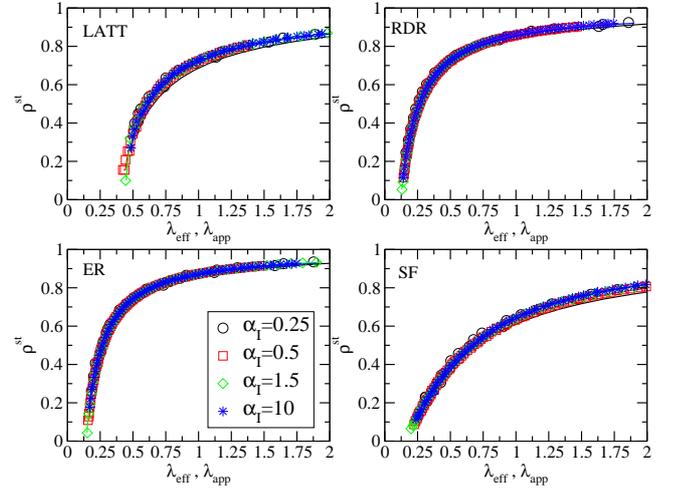

Figure 1: Steady-state prevalence ρ^{st} as a function of the effective infection rate, for different values of the exponent α_I. Symbols represent λ_{eff}, extracted by numerical simulations, continuous lines represent λ_{app}.

lapse onto one another when plotted as a function of λ_{eff} or λ_{app}. This result is particularly noteworthy since two infection processes with the same average infection time but different forms of $\psi_I(t)$ are known to behave very differently [3] showing huge differences in the prevalence ρ^{st}.

Interestingly, the approximate infection rate λ_{app} is expected to converge to λ_{eff} close to the epidemic threshold, and therefore the critical point and the set of critical exponents of the NM SIS dynamics, when expressed in terms of λ_{app}, are the same of those of the Markovian case. By means of a finite size scaling analysis, we obtain the epidemic threshold λ_c and the set of critical exponents for a NM SIS dynamics with $\alpha_I = 0.5$ and $\alpha_I = 2$, on top of two-dimensional lattice and RDR networks, showing that they are in very good agreement with the corresponding ones known in literature for Markovian SIS dynamics.

It is worth remarking that our formalism is not restricted to the SIS model and can be easily extended to any NM dynamics with a finite set of discrete states, allowing to determine the extent to which such dynamics can be reduced to a Markovian equivalent or whether the NM dynamics are fundamentally different. This simplification of the temporal nature of discrete-state processes promises to find application in the wide variety of areas where non-Markovian aspects are recognized as increasingly influential.

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