An anomalous diffusion equation for uniformly expanding media

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Anomalous diffusion plays an important role for many biological processes. Examples include chemical reactions leading to pattern formation in skin tissues and the development of embryonic tissues during morphogenesis [1]. In the latter case, the growth of the embryo results from an increase in the size of its cells followed by cellular division. Such a growth process can be regarded as an expansion of the cellular medium, which in turn provides a major motivation for the aim of the present work, i.e., the derivation of a novel Fokker-Planck (FP) equation describing intrinsic anomalous diffusion in an expanding medium. For the normal diffusion case, the corresponding FP equation was recently derived by Yuste et al. [2] from an extended Chapman-Kolmogorov equation.

Our starting point is a decoupled Continuous Time Random Walk (CTRW) model [4] in which particles perform instantaneous jumps. The jump length is given by a probability density function (PDF) $\lambda(y)$ with finite variance, whilst waiting times follow a heavy-tailed distribution $\varphi(t)$.

Transport in expanding media can be studied in two different frames of reference, respectively associated with physical or proper coordinates, $y(t)$, and comoving coordinates, $x(t)$. If the medium expansion is uniform, one has $y(t) = a(t)x(t)$, where $a(t)$ stands for the so-called scale factor. For the initial time, t_0, one has $a(t_0) = 1$.

A description in terms of comoving distances Δx turns out to be more convenient, despite the drawback of introducing an explicit time dependence of the jump length PDF. For example, a Gaussian distribution with variance equal to $2\sigma^2$ in physical space leads to the following PDF in comoving space:

$$\lambda(\Delta x|t) = \frac{1}{\sqrt{4\pi\sigma^2/a^2(t)}} \exp\left[-\frac{\Delta x^2}{4\sigma^2/a^2(t)}\right], \quad (1)$$

where $2\sigma^2/a^2(t)$ stands for the variance in the comoving frame of reference.

In Fourier-Laplace space, the waiting-time and jump displacement PDFs respectively behave as $\tilde{\varphi}(s) \sim 1 - \tau^\alpha s^\alpha$ or $\hat{\lambda}(k|t) \sim 1 - \sigma^2 k^2/a^2(t)$ in the diffusive limit (obtained by simultaneosly taking $s \to 0$ and $k \to 0$). Inserting the resulting expressions into the Montroll-Weiss equation and switching back to the space-time description, one finds

$$\frac{\partial}{\partial t} W(x,t) = \frac{K_\alpha}{a^2(t)} \, {}_0^{GL}D_t^{1-\alpha}\left[\frac{\partial^2}{\partial x^2} W(x,t)\right], \quad (2)$$

where $K_\alpha = \sigma^2/\tau^\alpha$ is the so-called anomalous diffusion coefficient. For sufficiently smooth functions, the Grünwald-Letnikov fractional derivative in the above equation can be replaced by the Riemann Liouville derivative [4],

$${}_0^{GL}D_t^{1-\alpha}\left[f(t)\right] = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t dt' \frac{f(t')}{(t-t')^{1-\alpha}}. \quad (3)$$

Solving the above integrodifferential equation does not seem straightforward, although some approximate solutions can be estimated. On the other hand, the comoving moments of the propagator can be computed from the hierarchy of equations

$$\frac{d}{dt}\langle x^2(t)\rangle = \frac{2K_\alpha t^{\alpha-1}}{\Gamma(\alpha)a^2(t)}, \quad (4)$$

$$\frac{d}{dt}\langle x^m(t)\rangle = m(m-1)\frac{2K_\alpha}{a^2(t)} \, {}_0^{GL}D_t^{1-\alpha}\left[\langle x^{m-2}(t)\rangle\right], \quad (5)$$

with $m = \{4, 6, 8...\}$. For example, for a power-law expansion $a(t) = [(t+t_0)/t_0]^\gamma$, the comoving second moment reads as follows:

$$\langle x^2(t)\rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)} t^\alpha \, {}_2F_1\left(\alpha, 2\gamma; 1+\alpha; -\frac{t}{t_0}\right). \quad (6)$$

The above analytic result for the second moment is in very good agreement with the one obtained from numerical simulations (see Fig. 1). We have also obtained similar equations for biased walks and for Lévy flights [3]. In this last case, the propagator is obtained analytically. Comparison with numerical results is provided.

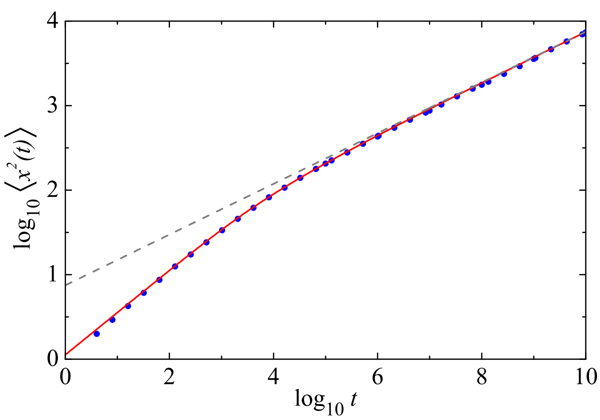

Figure 1: Log-log plot of the second moment in comoving coordinates versus time for a power-law medium expansion with $\gamma = 0.1$ and $t_0 = 1000$ for the anomalous diffusion exponent $\alpha = 0.5$. The solid line corresponds to the theoretical curve given by Eq. (6), the dots are simulation results, and the dashed line describes the long-time asymptotics.

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