Graphicality conditions for general scale-free complex networks and their aplication to visibility graphs.

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The great success of complex networks in explaining properties in many diverse phenomena is because networks provide these phenomena with a rich structure able to admit a quantitative analysis. In this way, relation phenomena occurring in such distinct fields as technology, sociology and biology [1] acquire a tangible structure amenable to analysis. Even other kinds of systems, such as time series that possess their own well defined and long established direct methods of analysis, are mapped to complex networks (visibility networks), with the aim of explaining phenomena not accessible to direct analysis [2]. Hence, the analysis and description of network structures is a subject of major interest. And the kind of network attracting most interest, since it is ubiquitously found in complex phenomena, is the so called scale-free network. Given a network with N nodes and some connections between them, defining k_i, the degree of each node i, as the number of edges supported by it, a scale-free network is defined as a graph whose degree distribution follows a power law $p_k \sim k^{-\gamma}$.

The exponent γ is, therefore, the main characterization of a scale-free network, but other asymptotic properties related with the large scale structure of networks are also important. In particular, the asymptotic behavior of the maximum degree k_N with the system size N follows also a power law, $k_N \sim N^{\kappa}$, and the exponent κ gives complementary information of the network structure. Due to the strong model dependence of κ, an analysis using both exponents γ and κ would be a good starting point in the characterization of a network structure. Studying how these exponents behave in particular models, or when some kind of structure exists, is a methodology followed in many papers. In general, one obtains structural bounds in the form of inequalities involving the exponents γ and κ. These bounds are dependent on the type of network. Here we are going to consider exclusively networks with neither multiple connections nor self connections.

The main result of this work [3] is to complete the present incomplete picture of the role played by bounds in scale-free complex networks. It is worth remarking that, for a general type of these networks, only the so-called natural bound, $\kappa \leq \frac{1}{\gamma-1}$, exists, and it applies in the range $\gamma > 2$. It is an effective bound in many kinds of networks, for instance, those growing with recursive methods which are, of course, correlated, and in general for networks in the range $\gamma \geq 3$. This bound is not due to structural constraints but it is an inherent property of any finite degree sequence with a power law distribution. Bounds due to structural constraints have been calculated, mainly for uncorrelated networks, using different methods: through properties of the degree-degree correlation [4], using statistical methods on network ensembles [5] or imposing graphicality conditions [6]. The structural bound for uncorrelated networks in the range $\gamma > 2$ is, from [4], $\kappa \leq 1/2$, whereas it is $\kappa < 1/\gamma$ in the range $1 < \gamma \leq 2$ [6]. Since natural and structural bounds arise from independent conditions, the exponent κ of the effective bound should be taken as $\min\{1/2, 1/(\gamma-1)\}$ in the range $\gamma > 2$. Note that for $\gamma \geq 3$ the effective bound is just the natural bound. To complete this picture in the range $\gamma > 2$ and taking as reference the bounds of uncorrelated networks, some papers deal with the effect of correlation on these bounds. Using phenomenological arguments [4] or, more quantitatively, with explicit models, it can be seen that a positive degree correlation (assortativity) produces a lower bound, whereas a negative degree correlation (disassortativity) increases the structural bound. Therefore, whereas in the range $\gamma > 2$ there is a complete picture of bounds, our knowledge in the range $1 < \gamma \leq 2$ is rather limited, only bounds for uncorrelated networks are known.

In this work we extend graphicality conditions to a general case of scale free networks, establishing that $\kappa < 1/\gamma$ is a general upper bound for networks with a well behaved power law in their degree distribution in the range $1 < \gamma \leq 2$. As a consequence of this result, and in order to show the importance of having this class of general results, we show how a recent estimation of exponents in power law distributions of visibility networks should be changed, since these conditions are not fulfilled.

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