The importance of being integrable: 30+1 years of the KPZ equation

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The Kardar-Parisi-Zhang (KPZ) equation describes the dynamics of a driven surface, subject to time-dependent fluctuations, and reads [1]

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta, \tag{1}$$

where $h(\mathbf{r}, t)$ is the height of the surface above point $\mathbf{r}$ on a d-dimensional reference plane at time t, the (Gaussian) noise $\eta(\mathbf{r}, t)$ has zero average and is uncorrelated in time and space, and $\nu > 0$ and λ are constant parameters.

Being deeply rooted in earlier studies of critical dynamics in the context of continuous phase transitions —notably for the case of randomly stirred fluids [2]—, the KPZ equation was put forward in 1986 as a continuum model of surface growth by irreversible aggregation. Agreement on scaling predictions with many discrete growth models led to establishing Eq. (1) as a paradigmatic system in the study of surface kinetic roughening or generic scale invariance. This was the case, in spite of a body of experimental evidence which, at the time, was surprisingly weaker than expected from theoretical considerations [3]. Intimate links with apparently very different systems, like polymers in random environments, further consolidated the KPZ equation as a sort of Ising model for Non-equilibrium Statistical Physics [4].

This talk will attempt a brief celebration of the time and achievements accumulated since the KPZ equation was initially put forward. A direct, timely motivation is the remarkable occurrence of breakthroughs that have taken place very recently in relation to this conspicuous model.

On the theoretical side, rigorous analytical results for discrete growth models in the KPZ universality class during the 2000's have culminated only in 2011 with the exact solution of the one-dimensional ($d = 1$) nonlinear KPZ equation, see references e.g. in [5]. These exact solutions show that the probability distribution of interface fluctuations is the Tracy-Widom (TW) distribution for the largest eigenvalue of Hermitian random matrices, well-known to describe non-Gaussian fluctuations as assessed in a number of contexts where random matrix theory is known to apply, like disordered conductors or, e.g., structural and spin glasses. A pictorial relation among these diverse fields is provided in Fig. 1, taken from the nice popularization article in [6].

Remarkably, in recent years the TW distribution —as well as the two-point correlation function obtained in the exact solutions of KPZ-related models— is being unambiguously identified in the dynamics of systems in the KPZ universality class, other than the KPZ equation itself. This *now includes experimental systems* like interfaces in turbulent liquid crystals or in colloidal systems. As a result, the KPZ equation is becoming a prime representative of a huge class of strongly-fluctuating one-dimensional systems that share the same scaling behavior, like classical nonlinear oscillators, stochastic hydrodynamics, quantum liquids, superfluids, or waves in reaction-diffusion systems. Furthermore, the KPZ plays a privileged role at this, illustrated by the existence of generalizations of the TW distributions that describe KPZ interfaces in 2+1 dimensions ($d = 2$), albeit at the price of the lack of exact forms [5]. It remains to be seen to which extent do the strong universality properties of the KPZ class depend upon the existence of an integrability structure. Again, a close analogy with the Ising model comes to mind: these two paradigms of Statistical Physics are exactly solvable in 1+1=2 dimensions. In analogy to the way in which the Ising model led to the elucidation of non-Gaussian fluctuations for critical equilibrium systems, one can still expect the KPZ equation to catalyze a deeper understanding of non-equilibrium fluctuations in years to come.

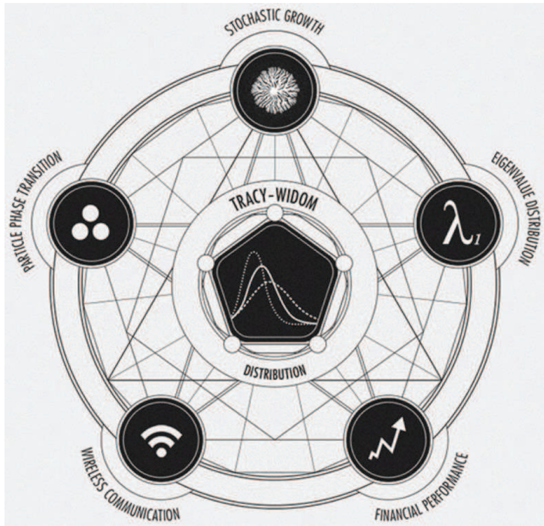

Figure 1: Skecth of relations between non-Gaussian random systems with Tracy-Widom-distributed fluctuations. Stochastic growth occurs via the KPZ universality class. By O. Shmahalo, as taken from [6].

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