Synchronization Transitions Induced by Topology and Dynamics

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The synchronization of coupled oscillators is a paradigmatic example of the emergence of complex behavior in a dynamical system with local interactions. It is an ubiquous phenomena in nature, where most of the systems present a complex underlying structure. Much research has been done in the last decades to understand the interplay between dynamics and topology, obtaining successful results in the inference of the structure from the response dynamics [1, 2] and the prediction of the synchronization onset for several topologies [3, 4, 5]. However, a general theory for synchronization in complex networks is still missing and there are many theoretical and empirical challenges to face towards a complete understanding of the process [6, 7].

In this work, we study the dynamics of Kuramoto oscillators in evolving complex topologies. We construct functionally equivalent networks by constraining the distribution of coupling strengths in the nodes in order to show that the same evolution of the global order parameter in a quasi-static process can be observed due to changes either in the underlying connectivity of the network or in the dynamics of the interactions. In this framework, an explicit analogy between topological and dynamic transitions is made by using simple mean-field arguments.

We consider that the dynamics of any sparse but connected network is driven by a reduced effective coupling strength between oscillators, K_{eff} depending only on the current coupling strength of the network K and its distribution over the nodes space. For instance, for an Erdös-Rényi $G(p, N)$, the homogenitiy of the network leads to the scaled coupling $K_{eff} = pK$, where p is the global fraction of existing edges, and the structural transition occurs at a critical connectivity $p_c = K_c/K, \forall K \geq K_c$ (K_c is the critical coupling for the all-to-all limit case). This result closely agrees with numerical simulations, and the mean-field approximation [4, 5] converges to it for large and highly connected systems. Beyond the prediction of the synchronization onset, we suggest that the whole evolution in the dynamic response due to structural changes is analogous to the evolution of an static structure under changes in the coupling strength among oscillators.

In order to quantify these effects, we use a model of network formation that interpolates between Erdös-Rényi and Scale-Free [8] to generate networks with increasing average connectivity constrained to the given degree distribution. For each network, we iterate the system towards the steady-state for a range of supercritical coupling strengths (for the all-to-all case), measuring the global degree of synchronization with the usual macroscopic order parameters.

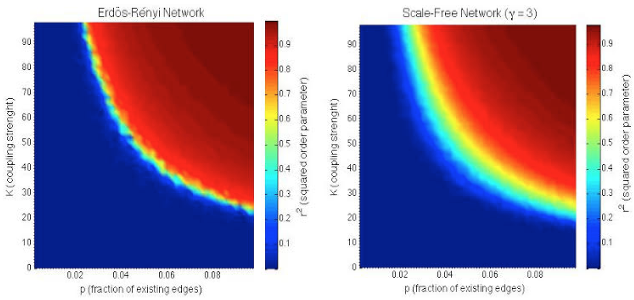

Figure 1: Steady-state of synchronization measured by the squared order parameter r^2 for both ER and SF networks in the plane (p, K), with $N = 10^3$ nodes and a uniform distribution of natural frequencies $g(\omega) = 1/\pi$, as fixed parameters in each realitzation. Isochrome regions represent functionally equivalent networks that preserve the global coupling strength and the clear analogy between structural (increasing p) and dynamic (increasing K) transitions is shown for both ensembles. For the ER case, we observe the same behaviour as in the all-to-all case, where a discontinuos phase transition occurs at $K_c = 2$, and the critical connectivity $p_c(K)$ matches our theoretical prediction. For SF networks, the transition appears earlier and becomes smoother, as expected [6, 7], but the studied analogy remains present.

This work presents some analytical and numerical evidence on the close relation between topological and dynamic transitions to synchronization. We aim to shed some light on the nature of these transitions in real systems, where one can usually measure their response dynamics, but there is very little information about the underlying topology, its evolution, and the specific local interaction mechanisms.

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